

# Essential norm estimates for weighted composition operator on the logarithmic Bloch space

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## Abstract

In this article, we estimate the essential norm of weighted composition operator  $W_{u,\varphi}$ , acting on the logarithmic Bloch space  $\mathcal{B}^{v_{\log}}$ , in terms of the  $n$ -power of the analytic function  $\varphi$  and the norm of the  $n$ -power of the identity function. Also, we estimate the essential norm of the weighted composition operator from  $\mathcal{B}^{v_{\log}}$  into the growth space  $H_{v_{\log}}^{\infty}$ . As a consequence of our result, we estimate the essential norm of the composition operator  $C_{\varphi}$  acting on the Logarithmic-Zygmund space.

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## 1 Introduction

Let  $\mathbb{D}$  be the unit disk of the complex plane  $\mathbb{C}$  and let  $H(\mathbb{D})$  be the space of all holomorphic functions on  $\mathbb{D}$  endowed with the topology of the uniform convergence on compact subsets of  $\mathbb{D}$ . For fixed holomorphic functions  $u : \mathbb{D} \rightarrow \mathbb{C}$  and  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ , we can define the linear operator  $W_{u,\varphi} : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  by

$$W_{u,\varphi}(f) := u \cdot (f \circ \varphi).$$

Which is known as the *weighted composition* with symbols  $u$  and  $\varphi$ . Clearly, if  $u \equiv 1$  we have  $W_{1,\varphi}(f) = f \circ \varphi = C_\varphi(f)$ , the composition operator  $C_\varphi$ , and if  $\varphi(z) = id(z) = z$  for all  $z \in \mathbb{D}$ , we have obtain  $W_{u,id}(f) = u \cdot f = M_u(f)$ , the multiplication operator  $M_u$ . Furthermore, we can see that  $W_{u,\varphi}$  is 1-1 on  $H(\mathbb{D})$  unless that  $u \equiv 0$  or  $\varphi$  is a constant function. However, if we wish to study properties like as continuity, compactness, essential norm, etc. of this operator, we need restrict the domain and target space  $H(\mathbb{D})$  to a normed and complete subspace of  $H(\mathbb{D})$ . In this article we consider the restriction to the growth space of all analytic functions  $f$  on  $\mathbb{D}$  such that

$$\|f\|_{H_v^\infty} = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty, \quad (1)$$

where  $v : \mathbb{D} \rightarrow \mathbb{R}^+$  is a *weight function*, that is, a bounded, continuous and positive function defined on  $\mathbb{D}$ . Also, we consider the Bloch-type space  $\mathcal{B}^v$  of all analytic function  $f$  on  $\mathbb{D}$  such that  $f' \in H_v^\infty$ . It is known that  $H_v^\infty$  is a Banach space with the norm defined in (1) and  $\mathcal{B}^v$  is a Banach space with the norm

$$\|f\|_{\mathcal{B}^v} = |f(0)| + \|f'\|_{H_v^\infty} = |f(0)| + \|f\|_{\tilde{\mathcal{B}}^v},$$

where,

$$\|f\|_{\tilde{\mathcal{B}}^v} = \sup_{z \in \mathbb{D}} v(z)|f'(z)|.$$

The properties of  $W_{u,\varphi}$  acting between growth-type spaces were studied by Hyvärinen et al. [7] and by Malavé-Ramírez and Ramos-Fernández [9], for very general weights  $v$ ; however, properties of  $W_{u,\varphi}$  acting on Bloch-type spaces are still in develops. About this last, we can mention the works of Hyvärinen and Lindström in [8]. Also, there are no much works about the properties of  $W_{u,\varphi}$  between  $H_v^\infty$  and  $\mathcal{B}^v$ , we can mention the work of Stević in [11].

In this note, we estimate the essential norm of  $W_{u,\varphi}$  acting on the logarithmic Bloch space  $\mathcal{B}^{v_{\log}}$  (also known as the weighted Bloch space), where the weight consider here is defined by

$$v_{\log}(z) = (1 - |z|) \log \left( \frac{2}{1 - |z|} \right)$$

with  $z \in \mathbb{D}$  which clearly is radial and typical ( $\lim_{|z| \rightarrow 1^-} v_{\log}(z) = 0$ ). This space appears in the literature when we study properties of certain operators acting on certain spaces of analytic functions on the unit disk  $\mathbb{D}$  of the complex plane  $\mathbb{C}$ . For instance, in 1991, Brown and Shields [2] showed that an analytic function  $u$  is a multiplier on the Bloch space  $\mathcal{B}$  if and only if  $u \in \mathcal{B}^{v_{\log}}$ . Also, in 1992, K. Attele [1]

showed that the Hankel operator induced by a function  $f \in H(\mathbb{D})$  in the Bergman space  $L_a^1$  (for the definition of Bergman space and the Hankel operator see [16]) is bounded if and only if  $\|f\|_{\mathcal{B}^{v_{\log}}} < \infty$ . The study of composition operators acting on the weighted Bloch space began with the work of Yoneda [15], where he characterized the continuity and compactness of composition operators acting on the weighted Bloch space  $\mathcal{B}^{v_{\log}}$ . These last results were extended by Galanoupulus [6] and Ye [13] for weighted composition operators acting on  $\mathcal{B}^{v_{\log}}$ . More recently, Malavé-Ramírez and Ramos-Fernández [9], following similar ideas used by Hyvärinen and Lindström in [8], characterized the continuity and compactness of  $W_{u,\varphi}$  acting on  $\mathcal{B}^{v_{\log}}$  in terms of certain expression involving the  $n$ -th power of  $\varphi$  and the log-Bloch norm of the  $n$ -th power of the identity function on  $\mathbb{D}$ . The results obtained by Malavé-Ramírez and Ramos-Fernández can be enunciated as follows:

**Theorem 1.1** ([9]). *Suppose that  $u : \mathbb{D} \rightarrow \mathbb{C}$  and  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  are holomorphic functions.*

1. *The operator  $W_{u,\varphi}$  is continuous on  $\mathcal{B}^{v_{\log}}$  if and only if*

$$\max \left\{ \sup_{n \in \mathbb{W}} \frac{(n+1) \|J_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}}, \sup_{n \in \mathbb{W}} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_n\|_{\mathcal{B}^{v_{\log}}}} \right\} < \infty.$$

2. *The operator  $W_{u,\varphi}$  is compact on  $\mathcal{B}^{v_{\log}}$  if and only if*

$$\max \left\{ \lim_{n \rightarrow \infty} \frac{(n+1) \|J_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}}, \lim_{n \rightarrow \infty} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_n\|_{\mathcal{B}^{v_{\log}}}} \right\} = 0,$$

where  $\mathbb{W} = \{0, 1, 2, \dots\}$ ,  $g_0 \equiv 1$ ,  $g_n(z) = z^n$  for  $n \in \mathbb{N} = \{1, 2, 3, \dots\}$  and  $z \in \mathbb{D}$ ,

$$w_{\log}(z) = \left[ \log \log \left( \frac{4}{1 - |z|^2} \right) \right]^{-1}, \quad (2)$$

and the functionals  $I_u, J_u : H(\mathbb{D}) \rightarrow \mathbb{C}$  are defined by

$$I_u(f(z)) = \int_0^z f'(s)u(s)ds, \text{ and } J_u(f(z)) = \int_0^z f(s)u'(s)ds.$$

The main goal of the present article is to find an estimation of the essential norm of the operator  $W_{u,\varphi} : \mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}$  which implies the result, about compactness, mentioned in the item (2) of Theorem 1.1 above. Allow us recall that the *essential norm* of a continuous operator  $T : X \rightarrow Y$ , between Banach spaces  $X$  and  $Y$ , denoted by  $\|T\|_e^{X \rightarrow Y}$ , is its distance to the class of the compact operators, that is,

$\|T\|_e^{X \rightarrow Y} = \inf\{\|T - K\|^{X \rightarrow Y} : K : X \rightarrow Y \text{ is compact}\}$ , where  $\|T\|^{X \rightarrow Y}$  denotes the norm of the operator  $T : X \rightarrow Y$ . Notice that  $T : X \rightarrow Y$  is compact if and only if  $\|T\|_e^{X \rightarrow Y} = 0$ .

Recent results about essential norm estimates on log-Bloch spaces can be found in [3] for the composition operator  $C_\varphi : \mathcal{B}^{v_3} \rightarrow \mathcal{B}^v$ , with  $v_3(z) = (1 - |z|) \log\left(\frac{3}{1-|z|}\right)$  and in [14], where Ye estimated the essential norm of the operator  $DC_\varphi : \mathcal{B}^{v_e} \rightarrow H_v^\infty$  defined by  $DC_\varphi(f) := W_{\varphi', \varphi}(f')$  with  $v_e(z) = (1 - |z|) \log\left(\frac{2e}{1-|z|}\right)$ .

In this article we are going to show the following result:

**Theorem 1.2.** *Suppose that  $u : \mathbb{D} \rightarrow \mathbb{C}$  and  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  are holomorphic functions and that  $W_{u, \varphi} : \mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}$  is continuous. Then*

$$\|W_{u, \varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} \simeq \max \left\{ \limsup_{n \rightarrow \infty} \frac{(n+1) \|J_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}}, \limsup_{n \rightarrow \infty} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_n\|_{\mathcal{B}^{v_{\log}}}} \right\}.$$

Above, and in what follows, for two positive quantities  $A$  and  $B$ , we write  $A \simeq B$  and say that  $A$  is equivalent to  $B$  if and only if there is a positive constant  $K$ , independent on  $A$  and  $B$ , such that  $\frac{1}{K} A \leq B \leq K A$ . To show Theorem 1.2, we establish, in Section 2, a triangle inequality which reduce our problem to estimate the essential norm of  $W_{u', \varphi} : \mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty$ . Such estimation is found in Section 3 in terms of the essential norm of  $W_{u, \varphi} : H_{v_{\log}}^\infty \rightarrow H_{w_{\log}}^\infty$ . Finally, in Section 4, we show Theorem 1.2. As a consequence of our results, in Section 5, we characterize continuity, compactness and we estimate the essential norm of the composition operator  $C_\varphi$  acting on the logarithmic-Zygmund space.

We want finish this introduction by mentioning that throughout this paper, constants are denoted by  $C$  or  $C_v$  (if depending only on  $v$ ), they are positive and may differ from one occurrence to the other.

## 2 A triangle inequality for the essential norm

Let  $\mu_1$  and  $\mu_2$  be two weights defined on  $\mathbb{D}$ . In this section we find upper bound for the essential norm of the operator  $W_{u, \varphi} : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  in terms of the essential norm of the operators  $W_{u', \varphi} : \mathcal{B}^{\mu_1} \rightarrow H_{\mu_2}^\infty$  and  $W_{u\varphi', \varphi} : H_{\mu_1}^\infty \rightarrow H_{\mu_2}^\infty$ . To this end, for a weight  $\mu$  we set the class

$$\tilde{\mathcal{B}}^\mu = \{f \in \mathcal{B}^\mu : f(0) = 0\},$$

which is a closed subspace of  $\mathcal{B}^\mu$ . With this notation, we have the following result:

**Lemma 2.1.** *If the operator  $W_{u,\varphi} : \tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is continuous, then*

$$\|W_{u,\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} \leq \|W_{u',\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow H_{\mu_2}^\infty} + \|W_{u\varphi',\varphi}\|_e^{H_{\mu_1}^\infty \rightarrow H_{\mu_2}^\infty}. \quad (3)$$

*Proof.* Let us suppose that the operator  $W_{u,\varphi} : \tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is continuous, then the continuous composition  $D_{\mu_2}W_{u,\varphi}D_{\mu_1}^{-1}$  maps the space  $H_{\mu_1}^\infty$  into  $H_{\mu_2}^\infty$ , where  $D_{\mu_1} : \tilde{\mathcal{B}}^{\mu_1} \rightarrow H_{\mu_1}^\infty$  and  $D_{\mu_2} : \tilde{\mathcal{B}}^{\mu_2} \rightarrow H_{\mu_2}^\infty$  denote the linear operators which transforms each  $f \in H(\mathbb{D})$  into its derivative  $f'$ . Clearly, the operators  $D_{\mu_1}$  and  $D_{\mu_2}$  are isometry and therefore they are invertibles with norms equal to 1. Furthermore, for every a  $f \in H_{\mu_1}^\infty$  we have the relation

$$D_{\mu_2}W_{u,\varphi}D_{\mu_1}^{-1}(f) = W_{u',\varphi}D_{\mu_1}^{-1}(f) + W_{u\varphi',\varphi}(f).$$

From the above relation, we deduce an expression for the operator  $W_{u,\varphi} : \tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  in terms of  $D_{\mu_1}$  and  $D_{\mu_2}$ , that is,

$$\begin{aligned} W_{u,\varphi} : \tilde{\mathcal{B}}^{\mu_1} &\rightarrow \mathcal{B}^{\mu_2} \\ W_{u,\varphi}(f) &= D_{\mu_2}^{-1}W_{u',\varphi}(f) + D_{\mu_2}^{-1}W_{u\varphi',\varphi}D_{\mu_1}(f). \end{aligned} \quad (4)$$

Now, by definition of essential norm, given  $\epsilon > 0$ , we can find compact operators  $\mathcal{K}_1 : \tilde{\mathcal{B}}^{\mu_1} \rightarrow H_{\mu_2}^\infty$  and  $\mathcal{K}_2 : H_{\mu_1}^\infty \rightarrow H_{\mu_2}^\infty$  such that

$$\begin{aligned} &\|W_{u',\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow H_{\mu_2}^\infty} + \|W_{u\varphi',\varphi}\|_e^{H_{\mu_1}^\infty \rightarrow H_{\mu_2}^\infty} \\ &\geq \frac{1}{1+\epsilon} \left( \|W_{u',\varphi} - \mathcal{K}_1\|^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow H_{\mu_2}^\infty} + \|W_{u\varphi',\varphi} - \mathcal{K}_2\|^{H_{\mu_1}^\infty \rightarrow H_{\mu_2}^\infty} \right). \end{aligned}$$

Thus, since the operators  $D_{\mu_1}$  and  $D_{\mu_2}$  are isometries, we can write

$$\begin{aligned} &\|W_{u',\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow H_{\mu_2}^\infty} + \|W_{u\varphi',\varphi}\|_e^{H_{\mu_1}^\infty \rightarrow H_{\mu_2}^\infty} \\ &\geq \frac{1}{1+\epsilon} \left( \|D_{\mu_2}^{-1}W_{u',\varphi} - D_{\mu_2}^{-1}\mathcal{K}_1\|^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} + \|D_{\mu_2}^{-1}W_{u\varphi',\varphi} - D_{\mu_2}^{-1}\mathcal{K}_2\|^{H_{\mu_1}^\infty \rightarrow \mathcal{B}^{\mu_2}} \right) \\ &\geq \frac{1}{1+\epsilon} \left( \|D_{\mu_2}^{-1}W_{u',\varphi} - D_{\mu_2}^{-1}\mathcal{K}_1\|^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} + \|D_{\mu_2}^{-1}W_{u\varphi',\varphi}D_{\mu_1} - D_{\mu_2}^{-1}\mathcal{K}_2D_{\mu_1}\|^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} \right) \\ &\geq \frac{1}{1+\epsilon} \|D_{\mu_2}^{-1}W_{u',\varphi} + D_{\mu_2}^{-1}W_{u\varphi',\varphi}D_{\mu_1} - (D_{\mu_2}^{-1}\mathcal{K}_1 + D_{\mu_2}^{-1}\mathcal{K}_2D_{\mu_1})\|^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} \\ &= \frac{1}{1+\epsilon} \|W_{u,\varphi} - \mathcal{K}\|^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} \geq \frac{1}{1+\epsilon} \|W_{u,\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}}, \end{aligned}$$

where we have used (4) in the last equality and the fact that the operator  $\mathcal{K} = D_{\mu_2}^{-1}\mathcal{K}_1 + D_{\mu_2}^{-1}\mathcal{K}_2D_{\mu_1} : \tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is compact since the composition of compact and continuous operators is a compact operator. The result follows because  $\epsilon > 0$  was arbitrary.  $\square$

Now, we show that the essential norm of the weighted composition operator, acting on Bloch-type spaces, does not change if we restrict the domain to  $\tilde{\mathcal{B}}^\mu$ .

**Lemma 2.2.** *If  $W_{u,\varphi} : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is continuous, then*

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} = \|W_{u,\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}}.$$

*Proof.* This proof uses similar arguments given in [5], (see Lemma 3.1). Since  $\tilde{\mathcal{B}}^{\mu_1} \subseteq \mathcal{B}^{\mu_1}$ , it is clear that all compact operator from  $\mathcal{B}^{\mu_1}$  into  $\mathcal{B}^{\mu_2}$  is also a compact operator from  $\tilde{\mathcal{B}}^{\mu_1}$  into  $\mathcal{B}^{\mu_2}$ ; hence we have

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} \geq \|W_{u,\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}}.$$

To show the reverse inequality, we observe that if  $T : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is any compact operator, then we can write

$$\begin{aligned} \|W_{u,\varphi} - T\|_e^{\mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} &= \sup_{\|f\|_{\mathcal{B}^{\mu_1}} \leq 1} \|W_{u,\varphi}(f) - T(f)\|_{\mathcal{B}^{\mu_2}} \\ &= \sup_{\|f\|_{\mathcal{B}^{\mu_1}} \leq 1} \|W_{u,\varphi}(f) + f(0)W_{u,\varphi}(\mathbf{1}) - f(0)W_{u,\varphi}(\mathbf{1}) - T(f - f(0)\mathbf{1} + f(0)\mathbf{1})\|_{\mathcal{B}^{\mu_2}} \\ &\leq \sup_{\|f\|_{\mathcal{B}^{\mu_1}} \leq 1} \|W_{u,\varphi}(f - f(0)\mathbf{1}) - T|_{\tilde{\mathcal{B}}^{\mu_1}}(f - f(0)\mathbf{1})\|_{\mathcal{B}^{\mu_2}} + \sup_{\|f\|_{\mathcal{B}^{\mu_1}} \leq 1} \|W_{u,\varphi}(f(0)\mathbf{1}) - T(f(0)\mathbf{1})\|_{\mathcal{B}^{\mu_2}} \\ &\leq \sup_{g \in \tilde{\mathcal{B}}^{\mu_1}, \|g\|_{\mathcal{B}^{\mu_1}} \leq 1} \|W_{u,\varphi}(g) - T|_{\tilde{\mathcal{B}}^{\mu_1}}(g)\|_{\mathcal{B}^{\mu_2}} + \sup_{h \in \mathcal{A}, \|h\|_{\mathcal{B}^{\mu_1}} \leq 1} \|W_{u,\varphi}(h) - T|_{\mathcal{A}}(h)\|_{\mathcal{B}^{\mu_2}} \\ &= \|W_{u,\varphi} - P_1\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} + \|W_{u,\varphi} - Q_1\|_e^{\mathcal{A} \rightarrow \mathcal{B}^{\mu_2}}, \end{aligned}$$

where  $\mathbf{1}(z) = 1$  for all  $z \in \mathbb{D}$ ,  $\mathcal{A}$  denotes the space of all constant functions in  $\mathcal{B}^{\mu_1}$ ,  $P_1 = T|_{\tilde{\mathcal{B}}^{\mu_1}}$  and  $Q_1 = T|_{\mathcal{A}}$ .

On the other hand, by definition of essential norm, given  $\epsilon > 0$ , we can find compact operators  $P : \tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  and  $Q : \mathcal{A} \rightarrow \mathcal{B}^{\mu_2}$  such that

$$\|W_{u,\varphi} - P\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} + \|W_{u,\varphi} - Q\|_e^{\mathcal{A} \rightarrow \mathcal{B}^{\mu_2}} \leq (1 + \epsilon) \left( \|W_{u,\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} + \|W_{u,\varphi}\|_e^{\mathcal{A} \rightarrow \mathcal{B}^{\mu_2}} \right).$$

But each  $f \in \mathcal{B}^{\mu_1}$  can be written as  $f = f(0) \cdot \mathbf{1} + g$ , where  $g \in \tilde{\mathcal{B}}^{\mu_1}$ . Furthermore, we can see that if  $f \in \mathcal{A}$  then  $g$  is the null function and if  $f \in \tilde{\mathcal{B}}^{\mu_1}$  then  $f = g$ . Thus, we can define the operator  $T : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  by

$$Tf = f(0)Q(\mathbf{1}) + P(g), \quad (f = f(0) \cdot \mathbf{1} + g \in \mathcal{B}^{\mu_1}).$$

Clearly,  $T$  is linear and compact operator from  $\mathcal{B}^{\mu_1}$  into  $\mathcal{B}^{\mu_2}$ . Indeed, if  $\{f_k\}$  is a bounded sequence in  $\mathcal{B}^{\mu_1}$ , then there exists a bounded sequence  $\{g_k\}$  in  $\tilde{\mathcal{B}}^{\mu_1}$  such that

$$f_k(z) = f_k(0) + g_k(z)$$

for all  $z \in \mathbb{D}$ . Bolzano-Weierstrass's theorem tell us that the numerical sequence  $\{f_k(0)\}$  has a convergent subsequence and since the operator  $P : \tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is compact, the sequence  $\{P(g_k)\}$  also has a convergent subsequence in  $\mathcal{B}^{\mu_2}$ . Hence  $\{T(f_k)\}$  has a convergent subsequence in  $\mathcal{B}^{\mu_2}$  and  $T : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is compact as was claimed.

Now, since  $T|_{\tilde{\mathcal{B}}^{\mu_1}} = P$  and  $T|_{\mathcal{A}} = Q$ . We obtain that

$$\begin{aligned} \|W_{u,\varphi}\|_e^{\mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} &\leq \|W_{u,\varphi} - T\|^{\mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} \\ &\leq \|W_{u,\varphi} - P\|^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} + \|W_{u,\varphi} - Q\|^{\mathcal{A} \rightarrow \mathcal{B}^{\mu_2}} \\ &\leq (1 + \epsilon) \left( \|W_{u,\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} + \|W_{u,\varphi}\|_e^{\mathcal{A} \rightarrow \mathcal{B}^{\mu_2}} \right). \end{aligned}$$

and since  $\epsilon > 0$  was arbitrary, we conclude that

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} \leq \|W_{u,\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} + \|W_{u,\varphi}\|_e^{\mathcal{A} \rightarrow \mathcal{B}^{\mu_2}}$$

and the result follows since the operator  $W_{u,\varphi} : \mathcal{A} \rightarrow \mathcal{B}^{\mu_2}$  is compact and therefore its essential norm is zero.  $\square$

As a consequence of Lemmas 2.1 and 2.2 we have the following result:

**Theorem 2.3.** *If the operator  $W_{u,\varphi} : \tilde{\mathcal{B}}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$  is continuous, then*

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}} \leq \|W_{u',\varphi}\|_e^{\tilde{\mathcal{B}}^{\mu_1} \rightarrow H_{\mu_2}^\infty} + \|W_{u\varphi',\varphi}\|_e^{H_{\mu_1}^\infty \rightarrow H_{\mu_2}^\infty}. \quad (5)$$

### 3 Estimation of the essential norm of $W_{u',\varphi} : \mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty$

From the conclusion of Theorem 2.3, we see that we have to estimate the essential norm of the operators  $W_{u\varphi',\varphi} : H_{\mu_1}^\infty \rightarrow H_{\mu_2}^\infty$  and  $W_{u',\varphi} : \tilde{\mathcal{B}}^{\mu_1} \rightarrow H_{\mu_2}^\infty$ . However, in the case of the weights  $v_{\log}$  and  $w_{\log}$ , the first one can be estimate using the results of Malavé-Ramírez and Ramos-Fernández in [9], since theses weights are radial and typical. Hence, in this section, we look for an upper bound for  $\|W_{u',\varphi}\|_e^{\tilde{\mathcal{B}}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty}$ . To this end, recall that for all  $f \in \mathcal{B}^{v_{\log}}$  the following relation holds:

$$|f(z)| \leq \left[ 1 + \log \log \left( \frac{2}{1 - |z|} \right) - \log \log(2) \right] \|f\|_{\mathcal{B}^{v_{\log}}}. \quad (6)$$

This fact allow us to show the following result, where  $w_{\log}$  is the weight defined in (2).

**Theorem 3.1.** *Suppose that  $u : \mathbb{D} \rightarrow \mathbb{C}$  and  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  are holomorphic functions and that the operator  $W_{u,\varphi} : \mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty$  is continuous. Then there exists a constant  $C > 0$  such that*

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty} \leq C \lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)|.$$

*Proof.* For each  $r \in (0, 1)$ , we consider  $K_r : \mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}$  defined by  $K_r(f) = f_r$ , where  $f_r$  is the dilatation of  $f$  given by  $f_r(z) = f(rz)$ . It is known that (see [3]) for each  $r \in (0, 1)$  the operator  $K_r$  is continuous and compact on  $\mathcal{B}^{v_{\log}}$ .

Let  $\{r_n\}$  any sequence in  $(0, 1)$  and consider the compact operators  $K_n : \mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}$  given by  $K_n = K_{r_n}$ , then the operators  $W_{u,\varphi}K_n : \mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty$  also are compacts for every  $n \in \mathbb{N}$ , since  $W_{u,\varphi} : \mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty$  is a continuous operator. Hence, by definition of essential norm we can write

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty} \leq \limsup_{n \rightarrow \infty} \|W_{u,\varphi} - W_{u,\varphi}K_n\|^{\mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty}.$$

Furthermore, for any  $f \in \mathcal{B}^{v_{\log}}$  such that  $\|f\|_{\mathcal{B}^{v_{\log}}} \leq 1$ , we have

$$\begin{aligned} \|(W_{u,\varphi} - W_{u,\varphi}K_n)(f)\|_{H_{v_{\log}}^\infty} &= \|u(f - f_{r_n}) \circ \varphi\|_{H_{v_{\log}}^\infty} \\ &= \sup_{z \in \mathbb{D}} v_{\log}(z) |u(z)f(\varphi(z)) - u(z)f(r_n\varphi(z))| \\ &= \sup_{z \in \mathbb{D}} v_{\log}(z) |f(\varphi(z)) - f(r_n\varphi(z))| |u(z)|. \end{aligned}$$

Now, we fix  $N \in \mathbb{N}$  and consider, for  $z \in \mathbb{D}$ , the cases  $|\varphi(z)| \leq r_N$  and  $|\varphi(z)| > r_N$ .

**Case 1:**  $|\varphi(z)| \leq r_N$

Since the operator  $W_{u,\varphi} : \mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty$  is continuous, then  $u = W_{u,\varphi}(\mathbf{1}) \in H_{v_{\log}}^\infty$ . Hence

$$\begin{aligned} \sup_{|\varphi(z)| \leq r_N} v_{\log}(z) |f(\varphi(z)) - f(r_n\varphi(z))| |u(z)| &\leq \|u\|_{H_{v_{\log}}^\infty} \sup_{|\varphi(z)| \leq r_N} |f(\varphi(z)) - f(r_n\varphi(z))| \\ &\leq \|u\|_{H_{v_{\log}}^\infty} \sup_{|w| \leq r_N} |f(w) - f_{r_n}(w)| \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ , since is a known fact that  $f_r \rightarrow f$  uniformly on compact subsets of  $\mathbb{D}$  as  $r \rightarrow 1^-$ .

**Case 2:**  $|\varphi(z)| > r_N$

By triangle inequality, we have

$$\begin{aligned} \sup_{|\varphi(z)| > r_N} v_{\log}(z) |f(\varphi(z)) - f(r_n\varphi(z))| |u(z)| \\ \leq \sup_{|\varphi(z)| > r_N} v_{\log}(z) |f(\varphi(z))| |u(z)| + \sup_{|\varphi(z)| > r_N} v_{\log}(z) |f(r_n\varphi(z))| |u(z)|. \end{aligned}$$



Thus, it is enough to find upper bounds for the expression in the right side of the above inequality. Put

$$\begin{aligned} L_1 &= \sup_{|\varphi(z)| > r_N} v_{\log}(z) |f(\varphi(z))| |u'(z)| \\ L_2 &= \sup_{|\varphi(z)| > r_N} v_{\log}(z) |f(r_n \varphi(z))| |u'(z)|. \end{aligned}$$

By the inequality (6), the fact that  $\|f\|_{\mathcal{B}^{v_{\log}}} \leq 1$  and since  $\log \log \left( \frac{2}{1-|\varphi(z)|} \right) \leq \log \log \left( \frac{4}{1-|\varphi(z)|^2} \right)$ , we obtain

$$\begin{aligned} L_1 &\leq \sup_{|\varphi(z)| > r_N} v_{\log}(z) |u(z)| \left[ 1 + \log \log \left( \frac{2}{1-|\varphi(z)|} \right) - \log \log(2) \right] \\ &\leq \sup_{|\varphi(z)| > r_N} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)| \left[ \frac{1 + \log \log \left( \frac{2}{1-|\varphi(z)|} \right) - \log \log(2)}{\log \log \left( \frac{4}{1-|\varphi(z)|^2} \right)} \right] \\ &\leq \sup_{|\varphi(z)| > r_N} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)| [w_{\log}(\varphi(z)) - \log \log(2) w_{\log}(\varphi(z)) + 1] \\ &\leq \sup_{|\varphi(z)| > r_N} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)| [w_{\log}(0) - \log \log(2) w_{\log}(0) + 1] \\ &\leq \sup_{|\varphi(z)| > r_N} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)| [2w_{\log}(0) + 1] \\ &\leq C \sup_{|\varphi(z)| > r_N} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)|, \end{aligned}$$

where  $C = 2w_{\log}(0) + 1 > 0$  and we have used that  $w_{\log}(r)$  is decreasing on  $[0, 1]$ .

In a similar way, we have that

$$L_2 \leq \sup_{|\varphi(z)| > r_N} \frac{v_{\log}(z)}{w_{\log}(r_n \varphi(z))} |u(z)| [2w_{\log}(0) + 1] \leq \sup_{|\varphi(z)| > r_N} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)| [2w_{\log}(0) + 1].$$

Hence, we can say that there exists a constant  $C > 0$  such that

$$\|(W_{u,\varphi} - W_{u,\varphi} K_n)(f)\|_{H_{v_{\log}}^\infty} \leq C \sup_{|\varphi(z)| > r_N} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)|.$$

Therefore, taking  $N \rightarrow \infty$  we have  $r_N \rightarrow 1^-$ ,

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty} \leq C \lim_{r_N \rightarrow 1^-} \sup_{|\varphi(z)| > r_N} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)|.$$

This shows the result.  $\square$

As a consequence of the above result we have the following estimation:

**Corollary 3.2.** *Suppose that  $u : \mathbb{D} \rightarrow \mathbb{C}$  and  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  are holomorphic functions and that the operator  $W_{u,\varphi} : \mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty$  is continuous. Then there exists a constant  $C > 0$  such that*

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty} \leq C \limsup_{n \rightarrow \infty} \frac{\|u\varphi^n\|_{H_{v_{\log}}^\infty}}{\|g_n\|_{H_{w_{\log}}^\infty}},$$

where  $g_n$  are the functions defined in Theorem 1.1.

*Proof.* Clearly, the weight  $w_{\log}$  is radial, typical and decreasing on  $(0, 1)$  and hence (see [7] or [9]), this weight is essential, that is,  $w_{\log} \simeq \tilde{w}_{\log}$ , where, for a weight  $v$ ,  $\tilde{v}$  denotes its associated weight given by

$$\tilde{v}(z) = \left( \sup_{\|f\|_{H_v^\infty} \leq 1} |f(z)| \right)^{-1}$$

with  $z \in \mathbb{D}$ . Also (see [4]), it is easy to see that  $v_{\log} \simeq v_3$ , where  $v_3$  is given by

$$v_3(z) = (1 - |z|) \log \left( \frac{3}{1 - |z|} \right) \quad (7)$$

with  $z \in \mathbb{D}$ . Thus,  $H_{v_{\log}}^\infty$  is equal to  $H_{v_3}^\infty$  with norms equivalents. Hence, from Theorem 3.1 we can write

$$\begin{aligned} \|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty} &\leq C \lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)| \\ &\leq C \lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} \frac{v_3(z)}{\tilde{w}_{\log}(\varphi(z))} |u(z)| \\ &= C \|W_{u,\varphi}\|_e^{H_{w_{\log}}^\infty \rightarrow H_{v_3}^\infty} \\ &\leq C \limsup_{n \rightarrow \infty} \frac{\|u\varphi^n\|_{H_{v_{\log}}^\infty}}{\|g_n\|_{H_{w_{\log}}^\infty}}, \end{aligned}$$

where we have used Theorem 2.1 in [10] in the equality and the last inequality is due to Theorem 2.4 in [7] (see also [9], Theorem 4.3).  $\square$

Now, using the definition of the functions  $g_n$ , the definition of the functional  $J_u$ , given in Theorem 1.1, and the fact that  $\|f\|_{\tilde{g}^v} = \|f'\|_{H_v^\infty}$  we can conclude:

**Corollary 3.3.** *Suppose that  $u : \mathbb{D} \rightarrow \mathbb{C}$  and  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  are holomorphic functions and that the operator  $W_{u,\varphi} : \mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty$  is continuous. Then there exists a constant  $C > 0$  such that*

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty} \leq C \limsup_{n \rightarrow \infty} \frac{(n+1) \|J_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}}.$$

## 4 The essential norm of $W_{u,\varphi} : \mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}$ . Proof of Theorem 1.2

Now we can give the proof of Theorem 1.2. From Theorem 2.3 we have

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} \leq \|W_{u',\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty} + \|W_{u\varphi',\varphi}\|_e^{H_{v_{\log}}^\infty \rightarrow H_{v_{\log}}^\infty}.$$

Thus, by Corollary 3.3 and Theorem 4.4 in [9], we can find a constant  $C > 0$  such that

$$\begin{aligned} & \|W_{u',\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow H_{v_{\log}}^\infty} + \|W_{u\varphi',\varphi}\|_e^{H_{v_{\log}}^\infty \rightarrow H_{v_{\log}}^\infty} \\ & \leq C \left( \limsup_{n \rightarrow \infty} \frac{(n+1)\|J_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}} + \limsup_{n \rightarrow \infty} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_n\|_{\mathcal{B}^{v_{\log}}}} \right) \\ & \leq C \max \left\{ \limsup_{n \rightarrow \infty} \frac{(n+1)\|J_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}}, \limsup_{n \rightarrow \infty} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_n\|_{\mathcal{B}^{v_{\log}}}} \right\}. \end{aligned}$$

Now we go to give a lower bound for  $\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}}$ . To this end, let  $K : \mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}$  be a compact operator and let  $\{z_n\}_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{D}$  such that  $|\varphi(z_n)| \rightarrow 1^-$  as  $n \rightarrow \infty$ . The following sequence was defined in [13],

$$f_n(z) = \frac{3}{a_n} \left[ \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right) \right]^2 - \frac{2}{a_n^2} \left[ \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right) \right]^3,$$

where  $a_n = \log \log \left( \frac{4}{1 - |\varphi(z_n)|^2} \right)$ . In [13] the author shown that  $\{f_n\}$  is a bounded sequence in  $\mathcal{B}^{v_{\log}}$ , that is, there exists a constant  $M > 0$  such that  $\|f_n\|_{\mathcal{B}^{v_{\log}}} \leq M$  for all  $n \in \mathbb{N}$ . This sequence converges to zero uniformly on compact subsets of  $\mathbb{D}$ . The derivatives of  $f_n$  is given by

$$f'_n(z) = \frac{\frac{6\overline{\varphi(z_n)}}{a_n} \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right)}{(1 - \overline{\varphi(z_n)}z) \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right)} - \frac{\frac{6\overline{\varphi(z_n)}}{a_n^2} \left[ \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right) \right]^2}{(1 - \overline{\varphi(z_n)}z) \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right)}.$$

Hence, we have  $f'_n(\varphi(z_n)) = 0$ ,  $f_n(\varphi(z_n)) = a_n$  and

$$\begin{aligned} M\|W_{u,\varphi} - K\|_{\mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} & \geq \limsup_{n \rightarrow \infty} \|(W_{u,\varphi} - K)(f_n)\|_{\mathcal{B}^{v_{\log}}} \\ & \geq \limsup_{n \rightarrow \infty} \|W_{u,\varphi}(f_n)\|_{\mathcal{B}^{v_{\log}}} - \limsup_{n \rightarrow \infty} \|Kf_n\|_{\mathcal{B}^{v_{\log}}} \\ & = \limsup_{n \rightarrow \infty} \|W_{u,\varphi}(f_n)\|_{\mathcal{B}^{v_{\log}}}, \end{aligned}$$

where we have used the known fact (see [12]) that if  $K : \mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}$  is compact then

$$\lim_{n \rightarrow \infty} \|K f_n\|_{\mathcal{B}^{v_{\log}}} = 0$$

for all bounded sequence  $\{f_n\} \subset \mathcal{B}^{v_{\log}}$  converging to zero uniformly on compact subsets of  $\mathbb{D}$ .

On the other hand,

$$\begin{aligned} \|W_{u,\varphi}(f_n)\|_{\mathcal{B}^{v_{\log}}} &= \|u f_n(\varphi)\|_{\mathcal{B}^{v_{\log}}} \\ &\geq \limsup_{n \rightarrow \infty} v_{\log}(z_n) |u'(z_n) f_n(\varphi(z_n)) + u(z_n) \varphi'(z_n) f'_n(\varphi(z_n))| \\ &= \limsup_{n \rightarrow \infty} v_{\log}(z_n) |u'(z_n) f_n(\varphi(z_n))| \\ &= \limsup_{n \rightarrow \infty} v_{\log}(z_n) |u'(z_n)| \log \log \left( \frac{4}{1 - |\varphi(z_n)|^2} \right) \\ &= \limsup_{n \rightarrow \infty} \frac{v_{\log}(z_n)}{w_{\log}(\varphi(z_n))} |u'(z_n)| \geq C \limsup_{n \rightarrow \infty} \frac{v_{\log}(z_n)}{\tilde{w}_{\log}(\varphi(z_n))} |u'(z_n)|, \end{aligned}$$

where, in the last inequality, we have used that the weight  $w_{\log}$  is essential. Thus, since the sequence  $\{z_n\}$  such that  $|\varphi(z_n)| \rightarrow 1^-$  was arbitrary, we can deduce that

$$\begin{aligned} \|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} &\geq C \limsup_{|\varphi(z)| \rightarrow 1^-} \frac{v_{\log}(z)}{\tilde{w}_{\log}(\varphi(z))} |u'(z)| \tag{8} \\ &\geq C \|W_{u',\varphi}\|_e^{H_{w_{\log}}^\infty \rightarrow H_{v_{\log}}^\infty} \\ &= C \limsup_{n \rightarrow \infty} \frac{\|u' \varphi^n\|_{H_{v_{\log}}^\infty}}{\|g_n\|_{H_{w_{\log}}^\infty}} \\ &= C \limsup_{n \rightarrow \infty} \frac{(n+1) \|J_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}}, \end{aligned}$$

where we have used the argument in the proof of Corollary 3.2.

Now, we go to show that there exists a constant  $C > 0$  such that

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} \geq C \limsup_{n \rightarrow \infty} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_n\|_{\mathcal{B}^{v_{\log}}}}.$$

As before, we consider a sequence  $\{z_n\} \subset \mathbb{D}$  such that  $|\varphi(z_n)| \rightarrow 1^-$  and we define the functions

$$h_n(z) = \frac{1}{\varphi(z_n) a_n^2} \left[ \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)} z} \right) \right]^3 - \frac{1}{\varphi(z_n) a_n} \left[ \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)} z} \right) \right]^2.$$

Then, clearly  $h_n(\varphi(z_n)) = 0$  for all  $n \in \mathbb{N}$ ,  $h_n$  converges to zero uniformly on compact subsets of  $\mathbb{D}$ ,

$$h'_n(z) = \frac{\frac{3}{a_n^2} \left[ \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right) \right]^2}{(1 - \overline{\varphi(z_n)}z) \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right)} - \frac{\frac{2}{a_n} \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right)}{(1 - \overline{\varphi(z_n)}z) \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right)},$$

and hence

$$h'_n(\varphi(z_n)) = \left[ (1 - |\varphi(z_n)|^2) \log \left( \frac{4}{1 - |\varphi(z_n)|^2} \right) \right]^{-1}.$$

Furthermore,  $\{h_n\}$  is a bounded sequence in  $\mathcal{B}^{v_{\log}}$ ; that is, there exists a constant  $M > 0$  such that  $\|h_n\|_{\mathcal{B}^{v_{\log}}} \leq M$  for all  $n \in \mathbb{N}$ . Indeed,

$$\begin{aligned} \|h_n\|_{\mathcal{B}^{v_{\log}}} &= \sup_{z \in \mathbb{D}} v_{\log}(z) |h'_n(z)| \\ &\leq \sup_{z \in \mathbb{D}} \frac{v_{\log}(z) \left[ \frac{3}{a_n^2} \left| \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right) \right|^2 + \frac{2}{a_n} \left| \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right) \right| \right]}{|1 - \overline{\varphi(z_n)}z| \log \left( \frac{4}{|1 - \overline{\varphi(z_n)}z|} \right)} \quad (9) \end{aligned}$$

where we applied the triangle inequality and the fact that  $|\log(w)| \geq \log(|w|)$  for each  $w \in \mathbb{D}$ . Furthermore, since

$$\lim_{x \rightarrow 0^+} \frac{\sqrt{\log^2 \left( \sqrt{\log^2 \left( \frac{4}{x} \right) + 4\pi^2} \right) + 4\pi^2}}{\log \log \left( \frac{2}{x} \right)} = 1,$$

then we can deduce that there exists a constant  $M > 0$  such

$$\frac{3}{a_n^2} \left| \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right) \right|^2 + \frac{2}{a_n} \left| \log \log \left( \frac{4}{1 - \overline{\varphi(z_n)}z} \right) \right| \leq M$$

for all  $n \in \mathbb{N}$  and all  $z \in \mathbb{D}$ . Hence, it is enough to find upper bound for

$$\sup_{z \in \mathbb{D}} \frac{v_{\log}(z)}{|1 - \overline{\varphi(z_n)}z| \log \left( \frac{4}{|1 - \overline{\varphi(z_n)}z|} \right)}$$

for all  $n \in \mathbb{N}$ . Clearly, the above expression is uniformly bounded if  $\frac{2}{e} \leq |1 - \overline{\varphi(z_n)}z| < 2$ . Now, if  $|1 - \overline{\varphi(z_n)}z| < \frac{2}{e}$ , then since the function  $h_2(t) = t \log \left( \frac{2}{t} \right)$  is increasing on  $(0, \frac{2}{e})$ , we have

$$\frac{v_{\log}(z)}{|1 - \overline{\varphi(z_n)}z| \log \left( \frac{4}{|1 - \overline{\varphi(z_n)}z|} \right)} \leq \frac{h_2(1 - |z|)}{h_2(|1 - \overline{\varphi(z_n)}z|)} \leq 1.$$

Hence, Inequality (4) is bounded and there exists a constant  $L > 0$  such that  $\|h_n\|_{\mathcal{B}^{v_{\log}}} \leq L$  for all  $n \in \mathbb{N}$ . Thus, we have

$$\begin{aligned}
\|W_{u,\varphi}(h_n)\|_{\mathcal{B}^{v_{\log}}} &\geq \limsup_{n \rightarrow \infty} v_{\log}(z_n) \frac{|u(z_n)\varphi'(z_n)|}{(1 - |\varphi(z_n)|^2) \log \left( \frac{4}{1 - |\varphi(z_n)|^2} \right)} \\
&\geq \limsup_{n \rightarrow \infty} v_{\log}(z_n) \frac{|u(z_n)\varphi'(z_n)|}{2(1 - |\varphi(z_n)|) \log \left( \frac{4}{1 - |\varphi(z_n)|} \right)} \\
&\geq C \limsup_{n \rightarrow \infty} v_{\log}(z_n) \frac{|u(z_n)\varphi'(z_n)|}{(1 - |\varphi(z_n)|) \log \left( \frac{2}{1 - |\varphi(z_n)|} \right)} \\
&= \limsup_{n \rightarrow \infty} \frac{v_{\log}(z_n)}{v_{\log}(\varphi(z_n))} |u(z_n)\varphi'(z_n)| \\
&\geq C \limsup_{n \rightarrow \infty} \frac{v_3(z_n)}{v_3(\varphi(z_n))} |u(z_n)\varphi'(z_n)| \\
&\geq C \limsup_{n \rightarrow \infty} \frac{v_3(z_n)}{\tilde{v}_3(\varphi(z_n))} |u(z_n)\varphi'(z_n)|
\end{aligned}$$

where we have used the fact that  $v_{\log} \simeq v_3$  and that  $v_3$  is an essential weight ( $v_3$  is the weight defined in (7)).

Since the sequence  $\{z_n\}$  such that  $|\varphi(z_n)| \rightarrow 1^-$  was arbitrary, we obtain that there exists a constant  $C > 0$  such that

$$\begin{aligned}
\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} &\geq C \limsup_{|\varphi(z)| \rightarrow 1^-} \frac{v_3(z)}{\tilde{v}_3(\varphi(z))} |u(z)\varphi'(z)| \quad (10) \\
&\geq C \|W_{u\varphi',\varphi}\|_e^{H_{v_3}^\infty \rightarrow H_{v_3}^\infty} \\
&= C \limsup_{n \rightarrow \infty} \frac{\|u\varphi'\varphi^n\|_{H_{v_{\log}}^\infty}}{\|g_n\|_{H_{v_{\log}}^\infty}} \\
&= C \limsup_{n \rightarrow \infty} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_n\|_{\mathcal{B}^{v_{\log}}}}.
\end{aligned}$$

Therefore, from the inequalities (8) and (10), we can conclude that there exists a constant  $C > 0$  such that

$$\|W_{u,\varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} \geq C \max \left\{ \limsup_{n \rightarrow \infty} \frac{(n+1)\|J_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}}, \limsup_{n \rightarrow \infty} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_n\|_{\mathcal{B}^{v_{\log}}}} \right\}.$$

This finishes the proof of Theorem 1.2.

## 5 An application. Composition operators on Zygmund-Logarithmic space

As an application of our results, in this section we study continuity, compactness and we estimate the essential norm of composition operators acting on Zygmund-logarithmic space. Recall that the Zygmund-logarithmic space  $\mathcal{Z}^{v_{\log}}$ , consists of all holomorphic functions  $f \in H(\mathbb{D})$  such that  $f' \in \mathcal{B}^{v_{\log}}$ . More precisely,

$$\mathcal{Z}^{v_{\log}} := \{f \in H(\mathbb{D}) : \|f\|_{\tilde{\mathcal{Z}}^{v_{\log}}} = \sup_{z \in \mathbb{D}} v_{\log}(z) |f''(z)| < \infty\}$$

endowed with the norm  $\|f\|_{\mathcal{Z}^{v_{\log}}} := |f(0)| + |f'(0)| + \|f\|_{\tilde{\mathcal{Z}}^{v_{\log}}}$ ,  $\mathcal{Z}^{v_{\log}}$  is a Banach space.

As before, for a holomorphic function  $u : \mathbb{D} \rightarrow \mathbb{C}$ , we define the functionals

$$I'_u f(z) = \int_0^z I_u(f(s)) ds \quad \text{and} \quad J'_u f(z) = \int_0^z J_u(f(s)) ds,$$

where  $f \in H(\mathbb{D})$  and  $I_u, J_u$  are the functionals defined in Theorem 1.1. We have the relations

$$\begin{aligned} \|f\|_{\tilde{\mathcal{Z}}^{v_{\log}}} &= \|f'\|_{\tilde{\mathcal{B}}^{v_{\log}}} \\ \|C_\varphi(f)\|_{\tilde{\mathcal{Z}}^{v_{\log}}} &= \|\varphi' C_\varphi(f)\|_{\tilde{\mathcal{B}}^{v_{\log}}}. \end{aligned}$$

Hence, the operator  $C_\varphi : \mathcal{Z}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}$  is continuous if and only if the weighted composition operator  $W_{\varphi', \varphi}$  is continuous on  $\mathcal{B}^{v_{\log}}$ . Thus, by Theorem 1.1, and the relations

$$\sup_{n \in \mathbb{W}} \frac{(n+1) \|J'_{\varphi'}(\varphi^n)\|_{\mathcal{Z}^{v_{\log}}}}{\frac{\|g_{n+2}\|_{\mathcal{Z}^{w_{\log}}}}{n+2}} = \sup_{n \in \mathbb{W}} \frac{(n+1) \|J_{\varphi'}(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}} < \infty$$

and

$$\sup_{n \in \mathbb{N}} \frac{\|I'_{\varphi'}(\varphi^n)\|_{\mathcal{Z}^{v_{\log}}}}{\frac{\|g_{n+1}\|_{\mathcal{Z}^{v_{\log}}}}{n+1}} = \sup_{n \in \mathbb{N}} \frac{\|I_{\varphi'}(\varphi^n)\|_{\mathcal{B}^{v_{\log}}}}{\|g_n\|_{\mathcal{B}^{v_{\log}}}} < \infty.$$

we have the following result:

**Corollary 5.1.** *Suppose that  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  is a holomorphic function. The composition operator  $C_\varphi$  is continuous on  $\mathcal{Z}^{v_{\log}}$  if and only if*

$$\max \left\{ \sup_{n \in \mathbb{W}} \frac{(n+2)(n+1) \|J'_{\varphi'}(\varphi^n)\|_{\mathcal{Z}^{v_{\log}}}}{\|g_{n+2}\|_{\mathcal{Z}^{w_{\log}}}}, \sup_{n \in \mathbb{N}} \frac{(n+1) \|I'_{\varphi'}(\varphi^n)\|_{\mathcal{Z}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{Z}^{v_{\log}}}} \right\} < \infty.$$

Now, we go to estimate the essential norm of the continuous operator  $C_\varphi : \mathcal{Z}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}$ , in fact, we go to show the following relation:

$$\|C_\varphi\|_e^{\mathcal{Z}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} = \|W_{\varphi', \varphi}\|_e^{\mathcal{B}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}}. \quad (11)$$

The argument in the proof of Lemma 2.2 shows that

$$\|C_\varphi\|_e^{\mathcal{Z}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} = \|C_\varphi\|_e^{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}},$$

where  $\tilde{\mathcal{Z}}^{v_{\log}} = \{f \in \mathcal{Z}^{v_{\log}} : f(0) = f'(0) = 0\}$ . Hence, it is enough to show that

$$\|C_\varphi\|_e^{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} = \|W_{\varphi', \varphi}\|_e^{\tilde{\mathcal{B}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}}.$$

To see this, we proceed as in the proof of Lemma 2.1, that is, we consider the derivative operator  $D : \tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}$ . Since this operator is an isometry, we have the relation

$$C_\varphi(g) := D^{-1}W_{\varphi', \varphi}D(g), \quad (12)$$

for all  $g \in \tilde{\mathcal{Z}}^{v_{\log}}$ . Thus, for  $\epsilon > 0$  we can find a compact operator  $T : \tilde{\mathcal{B}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}$  such that

$$\begin{aligned} \|W_{\varphi', \varphi}\|_e^{\tilde{\mathcal{B}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} &\geq \frac{1}{1+\epsilon} \|W_{\varphi', \varphi} - T\|_{\tilde{\mathcal{B}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} \\ &\geq \frac{1}{1+\epsilon} \|D^{-1}W_{\varphi', \varphi} - D^{-1}T\|_{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} \\ &\geq \frac{1}{1+\epsilon} \|D^{-1}W_{\varphi', \varphi}D - D^{-1}TD\|_{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} \\ &= \frac{1}{1+\epsilon} \|C_\varphi - K\|_{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} \geq \frac{1}{1+\epsilon} \|C_\varphi\|_e^{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}}, \end{aligned}$$

where we have used that  $D$  is an isometry and the fact that  $K := D^{-1}TD$  is a compact operator on  $\tilde{\mathcal{Z}}^{v_{\log}}$ . Therefore, since  $\epsilon$  was arbitrary, we conclude

$$\|C_\varphi\|_e^{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} \leq \|W_{\varphi', \varphi}\|_e^{\tilde{\mathcal{B}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}}.$$

Similarly, given  $\epsilon > 0$  we can find a compact operator  $\hat{T} : \tilde{\mathcal{Z}}^{\mu_1} \rightarrow \mathcal{Z}^{\mu_2}$  such that

$$\|C_\varphi\|_e^{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} \geq \frac{1}{1+\epsilon} \|C_\varphi - \hat{T}\|_{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}}.$$

Thus, using the fact that  $D$  is an isometry and the relation (12), we obtain

$$\begin{aligned} \frac{1}{1+\epsilon} \|C_\varphi - \hat{T}\|_{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} &= \frac{1}{1+\epsilon} \|DC_\varphi - D\hat{T}\|_{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} \\ &\geq \frac{1}{1+\epsilon} \|DC_\varphi D^{-1} - D\hat{T}D^{-1}\|_{\tilde{\mathcal{B}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} \\ &= \frac{1}{1+\epsilon} \|W_{\varphi', \varphi} - K\|_{\tilde{\mathcal{B}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}} \geq \frac{1}{1+\epsilon} \|W_{\varphi', \varphi}\|_e^{\tilde{\mathcal{B}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}}. \end{aligned}$$



Hence, since  $\epsilon$  was arbitrary, we conclude

$$\|C_\varphi\|_e^{\tilde{\mathcal{Z}}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} = \|W_{\varphi', \varphi}\|_e^{\tilde{\mathcal{B}}^{v_{\log}} \rightarrow \mathcal{B}^{v_{\log}}}.$$

Therefore, we have shown the following result:

**Theorem 5.2.** *Suppose that  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  is a holomorphic function and that  $C_\varphi$  is continuous on  $\mathcal{Z}^{v_{\log}}$ . Then*

$$\|C_\varphi\|_e^{\mathcal{Z}^{v_{\log}} \rightarrow \mathcal{Z}^{v_{\log}}} \asymp \max \left\{ \limsup_{n \rightarrow \infty} \frac{(n+2)(n+1) \|J'_{\varphi'}(\varphi^n)\|_{\mathcal{Z}^{v_{\log}}}}{\|g_{n+2}\|_{\mathcal{Z}^{w_{\log}}}}, \limsup_{n \rightarrow \infty} \frac{(n+1) \|I'_{\varphi'}(\varphi^n)\|_{\mathcal{Z}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{Z}^{v_{\log}}}} \right\}$$

As an immediate consequence of the above result, we have the following corollary:

**Corollary 5.3.** *Suppose that  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  is a holomorphic function. The composition operator  $C_\varphi$  is compact on  $\mathcal{Z}^{v_{\log}}$  if and only if*

$$\max \left\{ \lim_{n \rightarrow \infty} \frac{(n+2)(n+1) \|J'_{\varphi'}(\varphi^n)\|_{\mathcal{Z}^{v_{\log}}}}{\|g_{n+2}\|_{\mathcal{Z}^{w_{\log}}}}, \lim_{n \rightarrow \infty} \frac{(n+1) \|I'_{\varphi'}(\varphi^n)\|_{\mathcal{Z}^{v_{\log}}}}{\|g_{n+1}\|_{\mathcal{Z}^{v_{\log}}}} \right\} = 0.$$

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